Null Surfaces and the Legendre Submanifolds

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Abstract

It is shown that the main variable Z of the Null Surface Formulation of GR is the generating function of a constrained Lagrange submanifold that lives on the energy surface $H = 0$ and that its level surfaces $Z = const.$ are Legendre submanifolds on that energy surface. By globally extending the Lagrange submanifold over T^*M one obtains a generalized generating function $\widehat{Z}$ (a generating family). Thus, the singularity structure of the wavefronts can be obtained by studying the projection map of the Legendre submanifolds to the configuration space. The behaviour of the variable Z at the caustic points is analysed. It is shown that except for Minkowski space a single function $Z(x, \zeta, \bar{\zeta})$ cannot generate the conformal structure of a radiative space-time.

I. INTRODUCTION

The study of null geodesics on a space-time plays an important role in general relativity. Gravitational lensing, either cosmological, weak or micro lensing, is the study of the behaviour of null rays on a curved space-time. Null geodesics are used to define the notion of singularities on a space-time, to define the null boundary of a spacetime representing compact objects, etc.

In the last several years a formalism has been developed where null surfaces play a dynamical role replacing the metric as a basic variable [1–4]. The goal of the Null Surface Formulation of GR, or NSF for short, is to introduce a new variable such that from its knowledge one can obtain **all** the conformal structure of the space-time. Field equations equivalent to the Einstein's equation then determine the dynamical evolution of those surfaces. By casting GR as a theory of surfaces rather than a

theory of fields the NSF provides a completely new point of view with emphasis on the geometrical character of the theory. The basic variable is a function $Z(x^a, \zeta, \bar{\zeta})$ with x^a representing points on the spacetime and $(\zeta, \bar{\zeta})$ parametrizing the sphere of null directions. At each point on the space-time the function Z satisfies

$$g^{ab}(x)\partial_a Z(x, \zeta, \bar{\zeta})\partial_b Z(x, \zeta, \bar{\zeta}) = 0, \tag{1}$$

and the level surfaces of this function, namely $Z = const.$ are null hypersurfaces on the space-time. The reader should be aware that the above construction is done at a local level and that in general it might not be possible to find a single function satisfying these conditions on the whole space-time. Weyl curvature induces self-intersections and caustics on null congruences. Thus, even if one locally obtains a smooth hypersurface, extending such surface along the generators of its null geodesics will fail to be smooth. This generalized null surface is called a *wavefront* and cannot be described as the level surface of a single function Z.

The dynamics of the NSF is imposed as field equations for Z and another scalar Ω (the conformal factor) which are equivalent to the vacuum Einstein equations. The **global regular** solutions (in a suitable way to be specified in the conclusions) of those equations yield a radiative space-time, i.e., a space-time representing self interacting gravitational radiation. We see once again that even from a dynamical point of view it is very important to study the global behaviour of this variable, namely the behaviour of wavefronts in General Relativity.

The purpose of this work is to provide a framework (at a kinematical level) to discuss global behaviour of the basic variable Z. We want to analyze the singularities of our variable. We want to see under what circumstances a single function Z suffices to construct the entire conformal structure, or how many different functions must be given to cover the space-time. We only consider a specific class of space-times, asymptotically flat space-times along future null directions. This class of space-times represents isolated sources that may emit gravitational radiation.

The idea is to study the global solutions of eq.(1). Note that Z can be thought as the action of the Hamilton-Jacobi equation for the Hamiltonian $H(x, p) = g^{ab}p_a p_b$.

Since equation (1) can be written as

$$H(x, \partial_a Z) = 0,$$

Z is the action of the time-independent Hamilton-Jacobi equation and the study of the unicity of the solution and its global properties can be carried out using the tools of analytical mechanics. It is worth mentioning that the study of the solutions of the Hamilton Jacobi equations led to the development of the theory of Lagrange submanifolds on cotangent bundles and the loss of unicity on the solutions is directly related with the singularities of the projection map of these submanifolds onto the configuration space [5,6].

In this paper we reintroduce our variable $\widehat{Z}$ as the generating family of a constrained Lagrange submanifold that lives on the energy surface $H = 0$ and show that its level surfaces are Legendre submanifolds. Thus, the singularity structure of the wavefronts can be obtained by studying the projection maps to the configuration space. We thus, define the caustic set as the points on the Lagrange or Legendre submanifold with singular projection and the projection of those points as the caustics. Since Lagrange and Legendre submanifolds are smooth surfaces in T^*M this work suggests that one can redefine our variable in a way that is free from the singularities and self-intersections that are naturally associated with characteristic wavefronts in GR.

In Section II we introduce the necessary mathematical background needed for this work. In this context we also prove that the hypersurfaces of a constrained Lagrange submanifold defined as the restriction of this Lagrange submanifold to the level surfaces of its generating family are Legendre submanifolds on the energy surface $H = const.$

In Section III we give a specific example to clarify certain results of the previous section which are technically involved. We show how to construct the space-time wavefronts if the generating family is given and conversely, how to reconstruct the generating family if several smooth pieces of the wavefront are available. Note that we have left aside the study of the caustic points on the wavefronts and the caustic

set of the Legendre submanifold that generate those points. This is done in the next section.

In Section IV we study the singularity structure of our variable Z and the main results are found. We show that the caustic points are obtained by choosing the points were $\eth\bar{\eth}Z$, the parameter space laplacian of Z, blows up. We also show that at those points $(Z, \eth Z, \bar{\eth} Z)$ remain finite whereas $\eth^2 Z$ either vanishes or diverges. Using available singularity theorems we find as a proposition that except for Minkowski space a single function $Z(x, \zeta, \bar{\zeta})$ cannot generate the conformal structure of a radiative space-time. Thus, in order to properly study the global behaviour of the main variable in the NSF one must abandon the idea of using a single function on the space-time and instead one has to think of our variable as a generating family $\widehat{Z}$ of a Lagrange submanifold on the cotangent bundle of the space-time. We close this work with some comments of how to deal with the dynamics of the new variable.

II. LAGRANGE AND LEGENDRE SUBMANIFOLDS.

In this section we review the notions of Lagrange and Legendre manifolds in a given cotangent bundle T^*M of an n-dimensional manifold M. This subject has been fully developed and applied to different fields in the past twenty years. Exhaustive treatises at a high mathematical level and/or with beautiful applications to different areas in physics can be found in the literature [5–9]. The review here presented is tailored to our particular needs and by no means can be considered as a substitute for the standard references in the field.

In subsections A and B, we present some definitions and introduce the concept of *constrained Lagrange submanifold* in order to reinterpret, in the next section, our variable $\widehat{Z}$ as the generating family of a Lagrange manifold. Moreover, we prove in proposition II.7 that the hypersurfaces of a constrained Lagrange submanifold defined as the restriction of the Lagrange submanifold to the level surfaces of its generating family are Legendre submanifolds on the energy surface $H = const.$

A. Lagrange manifolds

Recall that (P, ω) is a symplectic manifold if P is an even-dimensional differentiable manifold and ω is a closed nondegenerate differential 2-form on P. We consider a particular kind of submanifolds of P called *Lagrange manifolds*.

Definition II.1 *Let (P, ω) be a symplectic manifold of dim $P = 2n$, a manifold L smoothly embedded by a map $e : L \to P$ is called a Lagrange submanifold of P if the pull-back to L of the symplectic form ω on P by e vanishes on L*

$$e^*\omega = 0,$$

and L is of maximal possible dimension compatible with the symplectic structure ω, i.e. dim $L = n$.

For these submanifolds, we introduce functions called *generating functions* as follows.

Definition II.2 *Let (P, ω) be a symplectic manifold, L a Lagrange submanifold, and $e : L \to P$ an embedding. Since locally, $\omega = -d\kappa$, then $e^*\omega = -d(e^*\kappa) = 0$, so $e^*\kappa = dS$ for a function $S : L \to \mathbf{R}$ (locally defined). We call S a generating function for L.*

From now on we will restrict ourselves to a particular class of symplectic manifolds, the cotangent bundle of an n-dimensional manifold M, denoted by T^*M. This bundle can be assigned local coordinates (q^i, p_i) with (q^i) representing points of M and p_i the local coordinates of the covectors at the point (q^i). In these local coordinates the closed nondegenerate differential 2-form ω on T^*M can be written as $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$.

If L is a Lagrange submanifold of (T^*M, ω), then the projection map $\pi : T^*M \to M$ given by $\pi(q^i, p_i) = q^i$, induces a map $\overline{\pi} = \pi \circ e$ called the *Lagrange map*. The set of points where the rank of $\overline{\pi}^*$ drops are called the *singular set* and the image of this set is called the *caustic*.

Notice also that if $S : M \to \mathbf{R}$, then the graph of dS, given in local coordinates by

$$\left\{ (q^i, p_i) \in T^*M : p_i = \frac{\partial S}{\partial q^i} \right\}. \tag{2}$$

is a Lagrange submanifold, as can be easily verified. Then, S is the *generating function* of the Lagrange manifold and $\overline{\pi}$ is a diffeomorphism.

The converse is also true, if $\overline{\pi}$ is locally a diffeomorphism, then L is the graph of dS, where $S : M \to \mathbf{R}$. In this case S is only locally defined.

Now, consider the hamiltonian system (T^*M, ω, H), where $H : T^*M \to \mathbf{R}$ is a hamiltonian function. The Lagrange submanifolds that we are interested on are those that can be considered as submanifolds of the energy hypersurface $\hat{H}$ defined by $H = const$. We shall refer to them as *constrained Lagrange submanifolds*.

Definition II.3 *Let $\hat{L}$ be a Lagrange submanifold of T^*M and H a Hamiltonian function, we say that $\hat{L}$ is a constrained Lagrange manifold if $\hat{L} \subset \hat{H}$, where $\hat{H}$ is an energy surface.*

Constrained Lagrange manifolds have very interesting properties. They are invariant under the flow of the hamiltonian vector field X_H ([6], Proposition 5.3.32). This, can be easily proved using the fact that X_H is tangent to the hypersurface $H = const.$ and that $\hat{L}$ is of maximal dimension . Thus, X_H is tangent to $\hat{L}$ and the hamiltonian flow preserves $\hat{L}$.

If $\hat{L}$ is the graph of dS, where $S : M \to \mathbf{R}$, then its generating function S must satisfy the time-independent Hamilton-Jacobi equation

$$H\left(q^j, \frac{\partial S}{\partial q^i}\right) = const, \tag{3}$$

i.e., the generating function $S = S(q^i)$ is the action of the Hamilton-Jacobi equation. Conversely, a solution of (3) locally defines a constrained Lagrange manifold with a diffeomorphic projection to M.

However, in general (and in the problem we want to address) $\hat{L}$ will not be globally diffeomorphic to M. How do we handle this situation? Since $\hat{L}$ is a smooth

hypersurface on T^*M contained in the hamiltonian flow, at most points the Lagrange projection will be a diffeomorphism (since the rank of $\overline{\pi}^*$ cannot drop more than $n-2$ on a set of points of zero measure with respect to the topology of M). We thus identify three different regions in $\hat{L}$:

1. smooth open regions of $\hat{L}$ diffeomorphic to M, and thus local graphs of a single function S,

2. points of $\hat{L}$ where the normal to $\hat{L}$ is "horizontal" in T^*M, i.e., the singular set, which divides **1** from

3. smooth open regions of $\hat{L}$ that project down to the same open neighborhood of M (these points of M have more than one preimage). These regions are generated by several smooth functions S_i.

From the point of view of Hamilton Jacobi theory unique solutions to (3) yield regions **1**, multivalued solutions to (3) yield regions **3** and singularities of the solutions yield the singular set **2**.

Let us briefly analize how to construct the three different regions from the solutions to the time independent Hamilton-Jacobi equation.

Since the Hamiltonian flow is tangent to $\hat{L}$, we need to solve Hamilton's canonical equations in order to generate the Lagrange manifold. A beautiful method to solve those equations is to find a generating function of a canonical transformation such that in the new variables the hamiltonian is independent of the new variables.

We recall that a canonical transformation is a diffeomorphism in T^*M that preserves the simplectic structure. Denoting the new variables by (Q^i, P_j). It is then easy to show that

$$p_i \mathrm{d}q^i - P_i \mathrm{d}Q^i = \mathrm{d}\widehat{S}(q^i, Q^j). \tag{4}$$

i.e. the difference between the canonical 1-forms associated with the two coordinates is exact. The function $\widehat{S}$ is called *the generating function* of the canonical transformation. Note that in the context of Lagrange manifolds this function $\widehat{S}$ is not the

generating function of a Lagrange manifold, it is called *generating family*, and that in Catastrophe Theory its normal forms are called *universal unfoldings* [9]. Hence in what follows we shall refer to $\widehat{S}$ as the *generating family*.

If the transformation is such that the Hamiltonian becomes constant in the new variables, then the new Hamilton's equations are trivially solved, i.e. $P_j = \beta_j,\ Q^i = \alpha^i$, where (α^i, β_j) are arbitrary constants. Using (4) we get

$$-\frac{\partial \widehat{S}}{\partial Q^i} = \beta_i$$
$$\frac{\partial \widehat{S}}{\partial q^i} = p_i$$

and this yields the Hamiltonian flow in the variables (q^i, p_j).

The question is how to find this very specific generating family $\widehat{S}$. The answer comes from the Hamilton-Jacobi theorem. Given the differential equation

$$H\left(q^i, \frac{\partial S}{\partial q^j}\right) = const \tag{5}$$

the complete integral of this equation, $\widehat{S} = \widehat{S}(q^i, \alpha_i)$, with $\alpha_i,\ i = 1...n$, arbitrary constants, is a generating function of a canonical transformation. To see this we set $Q^i = \alpha^i$ and $P_i = \dfrac{\partial \widehat{S}}{\partial \alpha_i}$. Then, if $\left|\dfrac{\partial^2 \widehat{S}}{\partial \alpha_i \partial q^j}\right| \neq 0$, Hamilton's canonical equations can be solved by quadratures. (Jacobi's Theorem [5,10])

It is then clear that the set $\hat{L}$ defined by

$$\hat{L} = \left\{(Q^i, P_j) |\ P_j = \beta_j\right\}$$

is a Lagrange manifold for each β_j. In particular, we can choose $P_j = 0$, or equivalently

$$\frac{\partial \widehat{S}(q^i, \alpha_j)}{\partial \alpha_i} = 0.$$

Then, given $\widehat{S}(q^i, \alpha_l)$ a complete integral of (5), with α_l being arbitrary constants for $l = 1...n$, we can obtain $\hat{L}$ locally as the graph of dS by solving first, if it is posible, $\alpha_l = \alpha_l(q^i)$ from the equations

$$\frac{\partial \widehat{S}}{\partial \alpha_l} = 0 \tag{6}$$

and then defining $S(q^i) = \widehat{S}(q^i, \alpha_j(q^i))$. This can be guaranteed if the rank of the system (6) is $r = n$ in the variables α_l. The Lagrange submanifold $\hat{L}$ is described by setting

$$p_i = \frac{\partial \widehat{S}}{\partial q^i},$$

and the *Lagrange map* $\bar{\pi}$ is a diffeomorphism (i.e. the rank of $\bar{\pi}^* = n$).

If $r = k < n$ then there exists $\alpha_J = \alpha_J(q^i)$, for $J = 1...k$ and $\widehat{S} = \widehat{S}(q^i, \alpha_I)$, for $I = k+1...n$. In this case the rank of $\bar{\pi}^*$ drops at some points and this can be related to the presence of these parameters α_I, for $I = k+1...n$.

Then, the solution of (5), $\widehat{S}(q^i, \alpha_I)$, defines a Lagrange submanifold $\hat{L} \subset \hat{H}$ embedded into T^*M by setting:

$$p_i = \frac{\partial \widehat{S}}{\partial q^i} \qquad 1 \leq i \leq n, \tag{7}$$

and imposing the constraints

$$0 = \frac{\partial \widehat{S}}{\partial \alpha_I}. \tag{8}$$

Since the rank of (8) is $n-k$ in the q^I variables, then there exist $q^I = q^I(\alpha_I, q^J)$, and $\hat{L}$ and $\bar{\pi}(\hat{L})$ are parametrized by (α_I, q^J). The derivative $\bar{\pi}^*$ can be written as

$$\begin{pmatrix} \frac{\partial q^I}{\partial \alpha_I} & \frac{\partial q^I}{\partial q_J} \\ 0 & I \end{pmatrix},$$

where I is the identity matrix $k \times k$, therefore it is clear that the rank of $\bar{\pi}^* \geq k$ and it shall be strictly less than n when

$$\left| \frac{\partial q^I}{\partial \alpha_I} \right| = 0.$$

The set of singular points and thus the caustic set shall be isolated points, curves or in general a set of points of zero measure with respect to the topology of M, since the rank of $\bar{\pi}^*$ cannot drop more than $n-2$. This assertion can be easily understood since there are two vector in $T^*\hat{L}$ that can not vanish under the projection, one is X_H and the other is the dual of $d\widehat{S}$. Notice that we can nevertheless write $\alpha_I = \alpha_I(q^i)$ if we allow α_I to be multivalued functions. As a consequence we obtain multivalued generating functions S_i.

B. Legendre manifolds

Odd-dimensional manifolds do not admit a *symplectic structure*. The analogue of a symplectic structure for odd-dimensional manifolds is a *contact structure*.

Definition II.4 *A contact manifold is a pair $(\hat{P}, \hat{\omega})$, consisting of an odd-dimensional manifold $\hat{P}$ and a closed 2-form $\hat{\omega}$ of maximal rank on this manifold. An exact contact manifold $(\hat{P}, \hat{\kappa})$ consists of a $(2n-1)$-dimensional manifold $\hat{P}$ and a 1-form $\hat{\kappa}$ on $\hat{P}$ such that $\hat{\omega} = -d\hat{\kappa}$ is of maximal rank on $\hat{P}$.*

Moreover, we can define a submanifold analogous to a Lagrange manifold, N of $\hat{P}$ called a *Legendre submanifold*.

Definition II.5 *Let $(\hat{P}, \hat{\kappa})$ be a contact manifold of dimension $2n-1$, a $(n-1)$-dimensional manifold N such that*

$$\hat{e}^* \hat{\kappa} = 0,$$

with $\hat{e} : N \to \hat{P}$ an embedding, is called a Legendre submanifold of $\hat{P}$.

Now, consider the hamiltonian system (T^*M, ω, H), the next proposition ensures that we can find, in a natural way, a *contact submanifold of T^*M*.

Proposition II.6 *Let (T^*M, ω, H) be a Hamiltonian system and $\hat{H}$ a regular energy surface, defined by $H = const$. Then $(\hat{H}, i^*\omega)$ is a contact manifold, where $i : \hat{H} \to T^*M$ is an inclusion ([6], Proposition 5.1.7).*

Thus the Legendre submanifolds we will consider are those that are submanifolds of the contact manifold $(\hat{H}, i^*\omega)$. Moreover they are hypersurfaces of *constrained Lagrange manifold* of a given Hamiltonian system and the projection map $\pi : T^*M \to M$ will induce a map $\hat{\pi}$ defined as $\hat{\pi} = \pi \circ \hat{e}$ and called *Legendre map.*

The next proposition gives a description of this kind of manifolds.

Proposition II.7 *Let $\hat{L}$ be a constrained Lagrange submanifold of the Hamiltonian system (T^*M, ω, H) and $\widehat{S}$ its generating family, i.e. a solution of the time independent Hamilton-Jacobi equation . Then the hypersurface $\hat{N}$ of $\hat{L}$, defined as the restriction of $\hat{L}$ to $\widehat{S} = const$, is a Legendre submanifold of $\hat{H}$.*

Proof: Given a Hamiltonian system, the Proposition II.6 ensures that $(\hat{H}, i^*\kappa)$ is a contact manifold and since $\hat{L}$ is a constrained Lagrange manifold, the generating function $\widehat{S}(q^i, \alpha_I)$ of $\hat{L}$ satisfies $H(q^i, \partial_j S) = const.$ Then $\widehat{S}$ defines a Legendre submanifold $\hat{N}$ of $\hat{H}$ by setting

$$p_i = \frac{\partial \widehat{S}}{\partial q^i} \qquad 1 \leq i \leq n, \tag{9}$$

imposing the constraints

$$\widehat{S} = const. \quad \text{and} \quad \frac{\partial \widehat{S}}{\partial \alpha_I} = 0, \tag{10}$$

and requiring that the rank of (10) shall be $n - k + 1$ in the q^I variables.

Recall that (10) is an algebraic non-linear system of equations, then if we define the function $G = (\widehat{S}, \frac{\partial \widehat{S}}{\partial \alpha_I})$, a solution of (10) satisfies $G = \mathbf{0}$, that is, it belongs to the kernel of the map G. Therefore demanding that the rank of the derivative of G to be $n - k + 1$ in the variables q^I and in one of the q^J variables, the implicit function Theorem guarantees that $q^i = q^i(q^j, \alpha_I)$ for $i \in I + 1$ and $j \in J - 1$.

Observe that the Legendre manifold constructed in this way becomes an hypersurface of $\hat{L}$ and that both are submanifold of the energy surface $\hat{H}$.$\square$

The image of the Legendre map is called the *wavefront* and the image of the constrained Lagrange manifold can be considered as a wavefront family. As in the case of the Lagrange manifolds, the set of points where the rank of $\hat{\pi}^*$ drops are called the *singular set* and the image of this set is called the *caustic.* If the singular set of $\hat{L}$ is known then intersection of this set with $\widehat{S} =$ const. yields the singular set of the associated Legendre submanifold.

III. EXAMPLES: A TOY MODEL

To clarify the concepts of constrained Lagrange and Legendre submanifolds as its level surfaces we present in this section a specific example. Using a particular generating family we describe the singularity structure of the wavefront. We also show how many pieces of smooth wavefronts are available on the configuration space for a

given a generating family. (Conversely, the Legendre manifold can be reconstructed if several pieces of smooth wavefronts are given.) We leave aside the issue of caustic points and caustics on the wavefronts which are analyzed in the next section.

Since we are interested in null surfaces on a Lorentzian manifold (M, g_{ij}) as Legendre submanifolds of an energy surface in T^*M, we take $H = \frac{1}{2} g^{ij}(q^k) p_i p_j$ as our Hamiltonian function and consider the hypersurface $H = 0$.

For simplicity, assume that $n = 3$ and $g_{ij} = \text{diag}(1, -1, -1)$. Then the Hamilton-Jacobi equation associated with this H is

$$\left(\frac{\partial S}{\partial q_1}\right)^2 - \left(\frac{\partial S}{\partial q_2}\right)^2 - \left(\frac{\partial S}{\partial q_3}\right)^2 = 0, \tag{11}$$

i.e. the eikonal equation. The complete integral of this equation is

$$\widehat{S} = \alpha_0 + \sum_{i=1}^{3} \alpha_i q_i,$$

where α_i satisfy

$$\alpha_1{}^2 - \alpha_2{}^2 - \alpha_3{}^2 = 0.$$

and $\alpha_0 = \widehat{S}_{|q^i=0}$. Hence the complete solution can be written as

$$\widehat{S}(q^i, \alpha_I) = \widehat{S}(0, \alpha_I) + \alpha_1 q_1 + \alpha_1 \sqrt{1 - \left(\frac{\alpha_2}{\alpha_1}\right)^2} q_2 + \alpha_2 q_3,$$

with $I = 1, 2$. The appearence of caustics shall depend on the choise of the parameters α_I and $\widehat{S}(0, \alpha_I)$ or equivalently on the initial value of the integral curves of X_H. Choosing $\widehat{S}(0, \alpha_I) = F(\alpha_I)$, were F are the germs of the normal forms of the generating functions of Lagrange manifold [5,9], we obtain the generating family for the singularities type A_2, A_3 o A_4. For example a cusp can be obtained if we choose $\widehat{S}(0, \alpha_I) = -\dfrac{\alpha_1{}^4}{2}$ and $\alpha_2 = \dfrac{\alpha_1{}^2}{2}$. Note that the universal unfolding of a cusp is (see [9])

$$\widehat{S}(\alpha, x, y) = \pm\alpha^4 + x\alpha^2 + y\alpha$$

Then, the functions

$$\widehat{S}(q_1,q_2,q_3,\alpha) = -\frac{\alpha^4}{4} + q_1\alpha + \frac{1}{2}q_2\alpha(4-\alpha^2)^{\frac{1}{2}} + \frac{1}{2}q_3\alpha^2$$
$$\frac{\partial\widehat{S}}{\partial\alpha}(q_1,q_2,q_3,\alpha) = -\alpha^3 + q_1 + q_3\alpha + \frac{1}{2}q_2(4-\alpha^2)^{\frac{1}{2}} - \frac{1}{2}q_2\alpha^2(4-\alpha^2)^{-\frac{1}{2}}$$

define a *constrained Lagrange manifold.* Note that the Taylor expansion of $\widehat{S}(q_1,q_2,q_3,\alpha)$ around $\alpha = 0$ is precisely the universal unfolding given above. The explicit construction follows.

From the equation

$$\frac{\partial\widehat{S}}{\partial\alpha} = -\alpha^3 + q_1 + q_3\alpha + \frac{1}{2}q_2(4-\alpha^2)^{\frac{1}{2}} - \frac{1}{2}q_2\alpha^2(4-\alpha^2)^{-\frac{1}{2}} = 0, \tag{12}$$

we trivially obtain

$$q_1 = \alpha^3 - q_3\alpha - \frac{1}{2}q_2(4-\alpha^2)^{\frac{1}{2}} + \frac{1}{2}q_2\alpha^2(4-\alpha^2)^{-\frac{1}{2}},$$

and by (7) we write $p_1 = \alpha$, $p_2 = \dfrac{1}{2}\alpha(4-\alpha^2)^{\frac{1}{2}}$ and $p_3 = \dfrac{\alpha^2}{2}$.

The map $e : \mathbf{R}^3 \to T^*M$

$$e(q_2,q_3,\alpha) = \Big(\alpha^3 - q_3\alpha - \frac{1}{2}q_2(4-\alpha^2)^{\frac{1}{2}} + \frac{1}{2}q_2\alpha^2(4-\alpha^2)^{-\frac{1}{2}}, q_2, q_3, \\ \alpha, \frac{1}{2}\alpha(4-\alpha^2)^{\frac{1}{2}}, \frac{\alpha^2}{2}\Big)$$

is an embedding and since $\widehat{S}$ satisfies (11) this surface is in $H = 0$. The Lagrange map becomes

$$\bar{\pi}^i(q_2,q_3,\alpha) = (\alpha^3 - q_3\alpha - \frac{1}{2}q_2(4-\alpha^2)^{\frac{1}{2}} + \frac{1}{2}q_2\alpha^2(4-\alpha^2)^{-\frac{1}{2}}, q_2, q_3), \tag{13}$$

clearly in a neighbourhood of $\alpha = 0$

$$\bar{\pi}^i(q_2,q_3,\alpha) = (\alpha^3 - q_3\alpha - q_2(1 - \frac{3}{8}\alpha^2), q_2, q_3),$$

and this yields a cusp.

The map (13) is not a diffeomorphism, since the equation

$$J = \det(\bar{\pi}^*) = -3\alpha^2 + q_3 + q_2\alpha(4-\alpha^2)^{-\frac{3}{2}}(\alpha^2 - 6) = 0,$$

gives us the points where the map losses its rank. Solving this equation we obtain

$$\begin{aligned} q_3 &= f(q_2, \alpha) \\ &= 3\alpha^2 - q_2\alpha(4-\alpha^2)^{-\frac{3}{2}}(\alpha^2 - 6). \end{aligned}$$

Then the caustic set is the image of $e(q_2, f(q_2, \alpha), \alpha)$ and the caustic is the image of $\bar{\pi}^i(q_2, f(q_2, \alpha), \alpha)$. The last is shown in FIG 1.

Remark III.1 *Observe that the projection of the surface $J = 0$ into configuration space (i.e. the caustic) divides regions on which the Lagrange map is a diffeomorphism, for example the region given by $\alpha > 0, q_2 > 0$ and $q_3 < 0$, from other regions where the Lagrange map in not injective (more than one preimage). In the first region we may locally write $\alpha = \alpha(q^i)$ from equation (12) and the generating function $S : M \to \mathbf{R}$. On the other hand, in the regions of non-injectivity, we obtain more than one function α_i as solution of the equation (12) which in turn implies that we get several functions $S_i : M \to \mathbf{R}$.*

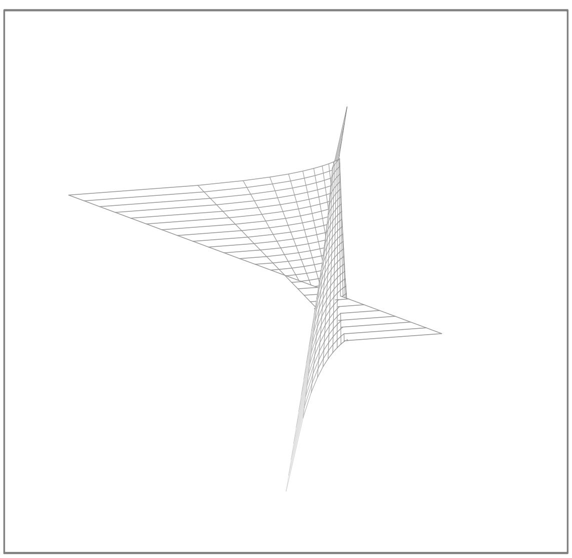

FIG. 1. The caustic

The corresponding Legendre submanifold is then constructed as a level surface of the Lagrange manifold presented above.

As before we write $p_1 = \alpha$, $p_2 = \dfrac{1}{2}\alpha(4-\alpha^2)^{\frac{1}{2}}$ and $p_3 = \dfrac{\alpha^2}{2}$. From the system

$$S = -\frac{\alpha^4}{4} + q_1\alpha + \frac{1}{2}q_3\alpha^2 + \frac{1}{2}q_2\alpha(4-\alpha^2)^{\frac{1}{2}} = 0$$

$$\frac{\partial S}{\partial \alpha} = -\alpha^3 + q_1 + q_3\alpha + \frac{1}{2}q_2(4-\alpha^2)^{\frac{1}{2}} - \frac{1}{2}q_2\alpha^2(4-\alpha^2)^{-\frac{1}{2}} = 0,$$

we obtain

$$q_1 = \frac{1}{2}\alpha^3 - 2q_2(4-\alpha^2)^{-\frac{1}{2}}, \quad q_3 = \frac{3}{2}\alpha^2 + q_2\alpha(4-\alpha^2)^{-\frac{1}{2}}$$

The map $\hat{e} : \mathbf{R}^2 \to T^*M$

$$\hat{e}(q_2,\alpha) = \Big(\frac{1}{2}\alpha^3 - 2q_2(4-\alpha^2)^{-\frac{1}{2}}, q_2, \frac{3}{2}\alpha^2 + q_2\alpha(4-\alpha^2)^{-\frac{1}{2}}, \alpha, \frac{1}{2}\alpha(4-\alpha^2)^{\frac{1}{2}}, \frac{\alpha^2}{2}\Big)$$

is an embedding and it defines a Legendre submanifold. The Legendre map

$$\hat{\pi}^i(q_2,\alpha) = (\frac{1}{2}\alpha^3 - 2q_2(4-\alpha^2)^{-\frac{1}{2}}, q_2, \frac{3}{2}\alpha^2 + q_2\alpha(4-\alpha^2)^{-\frac{1}{2}})$$

describes a wavefront, i.e. $\hat{\pi}(\hat{N})$. It is shown in FIG 2a.

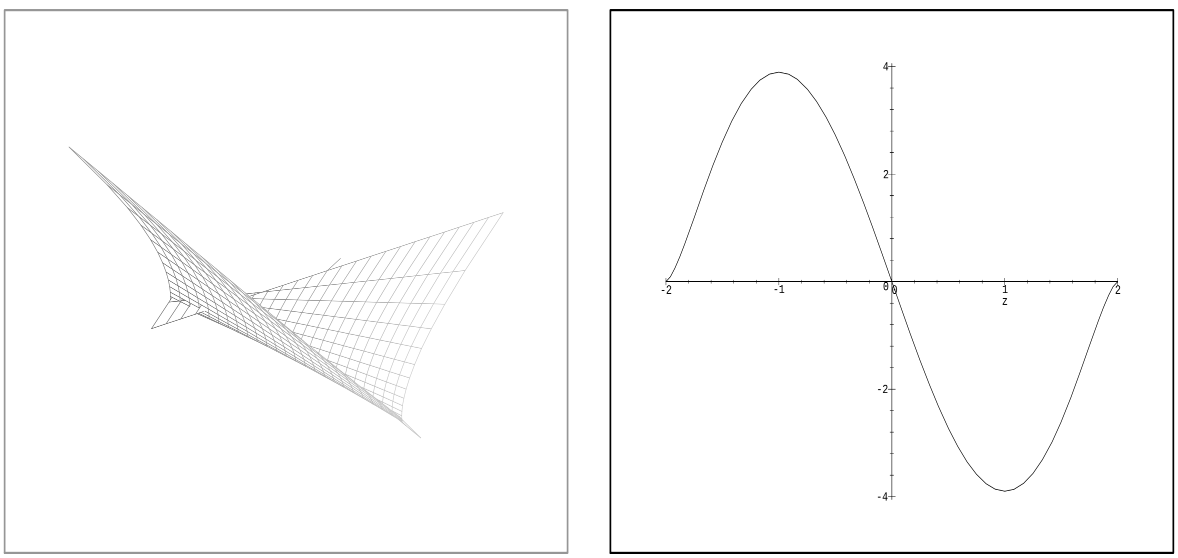

FIG. 2. (a) The wavefront and (b) $J = 0$

The derivative of the Legendre map is calculated as

$$\hat{\pi}^* = \begin{pmatrix} -\alpha f(\alpha,q_2) & -2(4-\alpha^2)^{-\frac{1}{2}} \\ 0 & 1 \\ 2f(\alpha,q_2) & \alpha(4-\alpha^2)^{-\frac{1}{2}} \end{pmatrix},$$

where $f(\alpha, q_2) = -\dfrac{4q_2 + 3\alpha(4-\alpha^2)^{\frac{3}{2}}}{2(4-s^2)^{\frac{3}{2}}}$. Clearly, the map $\hat{\pi}$ is not a diffeomorphism since the rank of its differential drops when

$$4q_2 + 3\alpha(4-\alpha^2)^{\frac{3}{2}} = 0,$$

i.e. on the curve $(\alpha, q_2(\alpha))$, $\alpha \in [-2, 2]$. This region (the caustic set) is depicted in FIG 2b. The caustic defined by these points (i.e. the image of the curve given above under $\hat{\pi}$) is given by

$$q_1 = -\alpha(\alpha^2-3), \qquad q_2 = -\frac{3\alpha}{4}(4-\alpha^2)^{\frac{3}{2}} \qquad \text{and} \qquad q_3 = \frac{3\alpha^2}{4}(\alpha^2-2), \quad \alpha \in [-2, 2].$$

and it is drawn in FIG 3a.

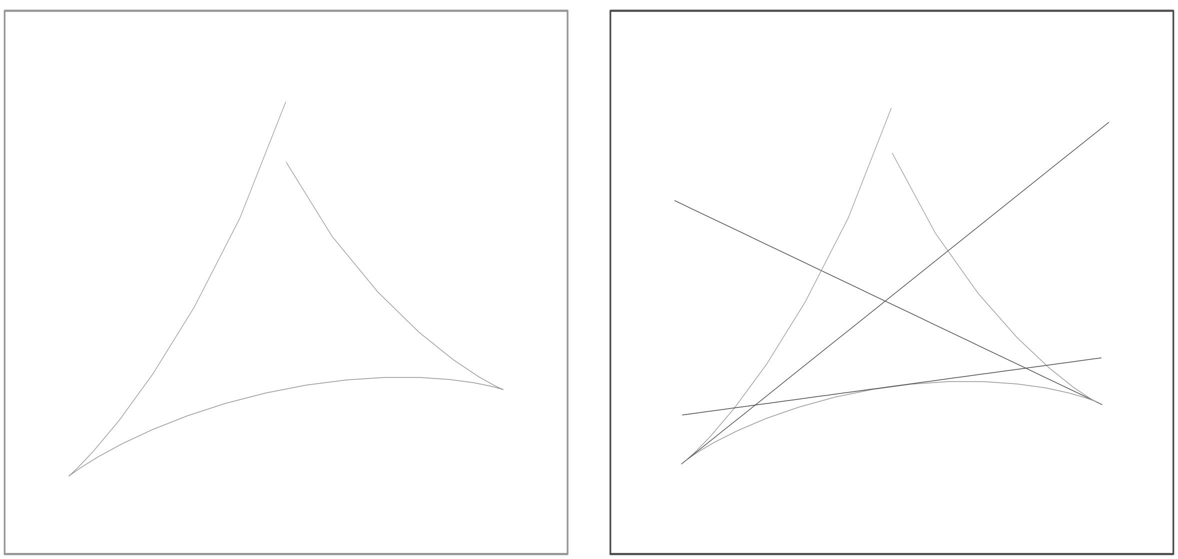

FIG. 3. (a) The caustic and (b) the null geodesics

This caustic, together with some integral curves of the non-vanishing null vector $l^i = \dfrac{\partial \hat{\pi}^i}{\partial q_2} = \left(-2(4-\alpha^2)^{-\frac{1}{2}}, 1, \alpha(4-\alpha^2)^{-\frac{1}{2}}\right)$ are shown in FIG 3b. Note that the caustic is the envelope of the null geodesics with tangent vector l^i and that the tangent vectors to the curves given $q_2 = \mathit{const}$ vanish on the caustic. The vanishing of these tangent vectors $M^i = \dfrac{\partial \hat{\pi}^i}{\partial \alpha}$ can be seen in FIG 4, where the caustic and some $q_2 = \mathit{const}$ curves (surfaces in higher dimensions) are shown.

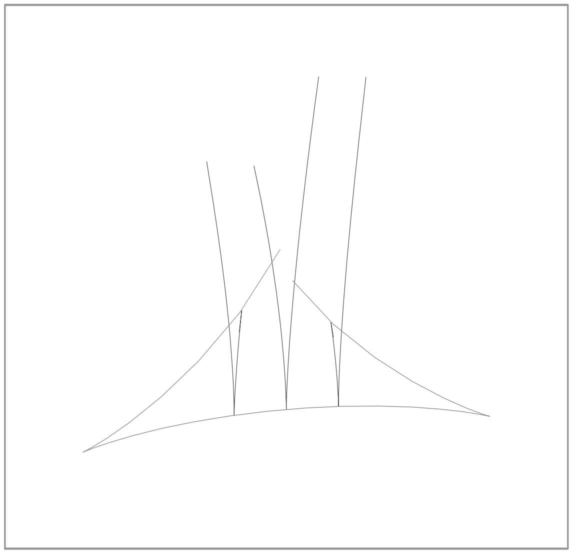

FIG. 4. The curves $q_2 = const$

Observe that in the region given by $-2 \leq \alpha \leq 2$ and $q_2 > 4$ the Legendre map is a diffeomorphism (see FIG 2). Thus, for a sufficiently large q_2, the function $\widehat{S}|_{q2=const}$ defines a submanifold $\hat{Q}$ diffeomorphic to $Q \approx \mathbf{R} \times [-A, A]$, being A a positive and large constant (see FIG 5).

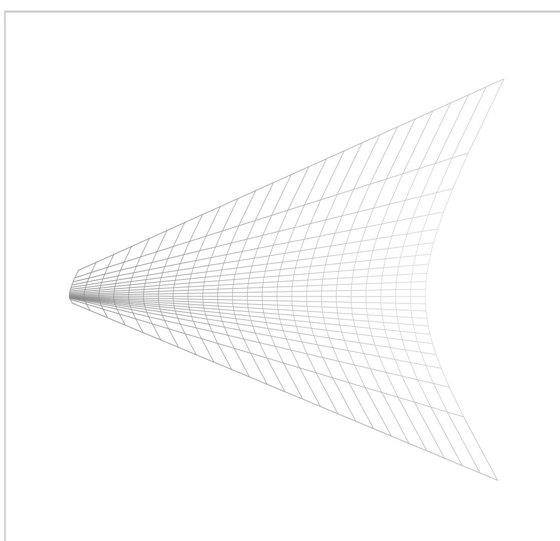

FIG. 5. A diffeomorphic region

From this example we conclude that:

Remark III.2 *Since there are regions outside the caustic set where the generating function* $S(q_i)$ *is multiple-valued, we have several pieces of a smooth wavefront. Conversely if we have several smooth pieces of a wavefront, using (9), we can reconstruct the Legendre manifold except at the caustic points.*

IV. THE FUNCTION Z IN ASYMPTOTICALLY FLAT SPACE-TIME

For simplicity we will consider asymptotically flat space times with a null boundary that represents the end points of future directed null geodesics [11]. Those space times represent compact objects that can emit gravitational radiation.

To define our variable Z we consider the intersection of the future null cone from x^a with the null boundary $\mathcal{I}^+$. Introducing Bondi coordinates $(u, \zeta, \bar{\zeta})$ on $\mathcal{I}^+$ (with u representing a Killing time and $(\zeta, \bar{\zeta})$ being stereographic coordinates on the unit sphere) this intersection is locally given by the equation

$$u = Z(x^a, \zeta, \bar{\zeta}). \tag{14}$$

Thus, for fixed values of x^a, the function Z yields the parametric description of the light cone cuts of $\mathcal{I}^+$.

On Minkowski spacetime the l.c. cuts adopt a very simple form, $u = x^a l_a$ with l_a a null covector constructed from the spherical harmonics $Y_{0,0}$, $Y_{1,-1}$, $Y_{1,0}$, $Y_{1,1}$.

In general a cut is a complicated surface with caustics, self-intersections, etc. However, it can be shown that for regular space-times the index number is always one. Therefore, a l.c. cut (as complicated as it might be) is always a continuous deformation of the sphere of null directions above each point x^a. It can also be shown that generically the l.c. cuts can only have two kinds of singularities, cusps and swallowtails since they represent the projection of 2-dim Legendre submanifolds on $\mathcal{I}^+$ [12].

A second meaning can be assigned to our variable $Z(x^a, \zeta, \bar{\zeta})$. Fixing a point $(u, \zeta, \bar{\zeta})$ of $\mathcal{I}^+$, the collection of interior points x^a that satisfy

$$Z(x^a, \zeta, \bar{\zeta}) = u = const, \tag{15}$$

form the past null cone of $(u,\zeta,\bar{\zeta})$. Moreover, from knowledge of Z we can construct a null coordinate system as follows.

Starting with our variable and taking $(\zeta,\bar{\zeta})$ derivatives of $Z(x^a,\zeta,\bar{\zeta})$ we construct the following set of scalars,

$$\theta^i(x^a,\zeta,\bar{\zeta}) \equiv (\theta^0,\theta^+,\theta^-,\theta^1) \equiv (u,\omega,\bar{\omega},R) \equiv (Z,\eth Z,\bar{\eth} Z,\eth\bar{\eth} Z). \tag{16}$$

For fixed values of $(\zeta,\bar{\zeta})$ they define a coordinate system with the following geometric meaning,

- u = const. denotes the past null cone from $(u,\zeta,\bar{\zeta})$.
- $(\omega,\bar{\omega})$ = const. single out a null geodesic on that surface.
- R = const. identifies a point on that geodesic.

However, one knows that null cones can develop caustics and singularities. One also knows that past those singularities the null cone is no longer smooth (it is called a wavefront) and thus, a null coordinate system like the one above breaks down past those singular points. Since the main goal of the NSF is to replace the metric with a function Z such that its level surfaces are past null cones from $\mathcal{I}^+$, we immediately face a non trivial problem: if the null cones develop self-intersections and singularities that cannot be analyzed with a single function Z then the construction given above is only valid on a neighborhood of $\mathcal{I}^+$. However, we also know that null wave fronts are projections of Legendre submanifolds that live on $T^*(M)$. It would then appear that the best way to deal with this lack of smoothness is to think of our variable as the generating family $\widehat{Z}$ of a constrained lagrange submanifold. An outline of this construction is presented below.

As was done before we assume that Z is a solution of the equation

$$H(x^a,\partial_b Z) = g^{ab} Z_{,a} Z_{,b} = 0. \tag{17}$$

with g^{ab} a metric that is asymptotically flat.

In a neighborhood of $\mathcal{I}^+$ the solution to this equation yields a single function Z and its level surfaces $Z = const.$ describe the past null cones from points at the null boundary. This follows from the fact that the (unphysical) metric near $\mathcal{I}^+$ is "almost" conformally flat and thus the past null cones are free from caustics and singularities.

Since $Z(x^a, \zeta_0, \bar{\zeta}_0)$ is a smooth function on this region, we can choose $S = Z(x^a, \zeta_0, \bar{\zeta}_0)$ as the generating function of a constrained Lagrange manifold $\hat{L}$

$$N = \left\{ (x^a, p_b = \frac{\partial Z}{\partial x^b}) : e^* \kappa = \mathrm{d}Z \right\}. \tag{18}$$

Note that the manifold described above is equivalent to (2) and the Proposition II.7 ensures that the surface $\hat{N}$ defined by $Z = const$ is a Legendre submanifold of the energy surface given by $H = 0$, i.e.

$$\hat{N} = \left\{ (x^a, p_b = \frac{\partial Z}{\partial x^b}) : \hat{e}^* \hat{\kappa} = \mathrm{d}(e^* Z) = 0 \right\}.$$

Thus, we have constructed a constrained Legendre submanifold $\hat{N}$ and a constrained Lagrange submanifold $\hat{L}$ of $\hat{H}$ using our fundamental variable $Z(x^a, \zeta_0, \bar{\zeta}_0)$.

The idea is to extend this construction to regions were caustics develop. As we mentioned before, in these regions the Lagrange submanifold is not diffeomorphic to its projection. We will thus assume that a solution of $H(x^a, \partial_b Z) = 0$ can be written as $\widehat{Z} = \widehat{Z}(x^a, \zeta_0, \bar{\zeta}_0, \alpha_I)$ with $I = 1, 2$. The function $\widehat{Z}$ depends, at most, on two parameters since the rank of the projection map cannot drop more than two [5]. The constrained Lagrange submanifold $\hat{L}$ as described in section II A is given by

$$p_a = \frac{\partial \widehat{Z}}{\partial x^a}.$$

together with the constraint

$$\frac{\partial \widehat{Z}}{\partial \alpha_I} = 0. \tag{19}$$

Observe that if we can solve (19) uniquely for α_I, i.e. $\alpha_I = \alpha_I(x^b)$, then $\widehat{Z} = Z$ and we are back in the previous diffeomorphic region. In general, one will obtain multi-valued solutions of (19). Inserting the different solutions of α_I into $\widehat{Z}(x^a, \zeta_0, \bar{\zeta}_0, \alpha_I)$

one obtains a multiple-valued function $Z(x^a, \zeta_0, \bar{\zeta}_0)$. The Legendre submanifold is obtained by setting $\widehat{Z} = const$. Conversely, if several functions Z_i are given, one can reconstruct the Lagrange submanifold by imposing (18) on the different Z's. The construction defines the lagrange submanifold except for the caustic set.

Finally, we would like to determine under what circumstances it is possible to find a single function Z that would yield for us a global coordinate system $(u, R, w, \bar{w})$ on an asymptotically flat space time. In other words we want to know if there exists space times that are diffeomorphic to the corresponding Lagrange manifolds. At the same time we would like to know when and how this coordinate system breaks down due to the presence of conjugate points. We are therefore interested in describing the relationship between our fundamental variable Z and the loss of the rank of the derivative of the Legendre map $\hat{\pi}$, i.e. we want to described the singular set in terms of Z.

When the Lagrange manifold is a constrained one, the loss of rank of the Lagrange map indicates the non-existence of global solutions of the Hamilton-Jacobi equation and the loss of rank of the associated Legendre map is related to the *existence of conjugate points of a congruence of null geodesics.*

In order to clarify this assertion, we consider the local description of the wavefront (the projection of the Legendre manifold). We assume that the wavefront is locally described by

$$x^a = f^a(u_0, s, w, \bar{w}, \zeta_0, \bar{\zeta}_0).$$

with s an affine length. The vectors

$$L^a = \frac{\partial f^a}{\partial s}, \qquad M^a = \frac{\partial f^a}{\partial w} \qquad \text{and} \qquad \bar{M}^a = \frac{\partial f^a}{\partial \bar{w}}.$$

are tangent to the wavefront. L^a is directed along the null geodesics whereas M^a and $\bar{M}^a$ are geodesic deviation vectors.

The derivative of the Legendre map losses its rank when these three vectors become linearly dependent. This dependence is related to the existence of conjugate points on the congruence of null geodesic with apex at $\mathcal{I}^+$ and null tangent vector L^a as follows.

We introduce the parallelly propagated null triad $\{l^a, m^a, \bar{m}^a\}$, satisfying

$$\left.\begin{aligned} l^a m_a \quad &= 0 \\ m^a \bar{m}_a \quad &= -1 \\ l^a \nabla_a m^b &= 0. \end{aligned}\right\} \tag{20}$$

In terms of this triad

$$L^a = l^a, \quad M^a = \xi m^a + \bar{\eta}\bar{m}^a, \quad \bar{M}^a = \bar{\xi}\bar{m}^a + \eta m^a, \tag{21}$$

therefore this set of vectors becomes linearly dependent when

$$\begin{vmatrix} \xi & \eta \\ \bar{\eta} & \bar{\xi} \end{vmatrix} = (\xi\bar{\xi} - \eta\bar{\eta}) = 0. \tag{22}$$

On the other hand, this quantity is related to the divergence ρ, and the shear σ of the congruence with apex in $\mathcal{I}^+$. To see this , consider the optical parameters

$$\rho = m^a \bar{m}^b \nabla_a l_b \qquad \text{and} \qquad \sigma = m^a m^b \nabla_a l_b.$$

Using eq.(21) together with the fact that M^a is Lie propagated along the null direction L^a we get([3])

$$\sigma = \frac{\bar{\eta}^2}{A} D\left(\frac{\bar{\xi}}{\bar{\eta}}\right) \qquad \rho = \frac{DA}{2A}$$

with $A = (\xi\bar{\xi} - \eta\bar{\eta})$ and where we have used the fact that ρ is real.

Hence, at the points where the Legendre map looses its rank, the divergence of the congruence becomes unbounded, i.e. $\lim \rho_{s \to s_0} = \infty$, where s is the affine parameter and s_0 corresponds to a conjugate point.

Summarizing, the loss of rank of the Legendre map, i.e. the development of a caustic is directly related to the existence of a conjugate point on the null congruence with apex at $\mathcal{I}^+$.

Since Z is a unique function near $\mathcal{I}^+$, $(u, R, w, \bar{w})$ is a well behaved coordinate system in that region. The question is what happens to our coordinates as we approach a generic conjugate point.

Lemma IV.1 *Let $Z(x^a, \zeta, \bar{\zeta})$ be our basic variable near $\mathcal{I}^+$ such that $Z = 0$ describes the past null cone from $(u, \zeta, \bar{\zeta})$. Then at the first conjugate point of this null congruence the scalar $R = \eth\bar{\eth}Z$ goes to $-\infty$.*

Proof: Given a null geodesic labeled by $(u, w, \bar{w}, \zeta, \bar{\zeta})$ we introduce an affine length s and two null congruences that contain this geodesic; 1) the future null cone with apex at s_0 and 2) the past null cone with apex at $(u, \zeta, \bar{\zeta})$. The divergence of the first congruence at $\mathcal{I}^+$ is related to the value of the scalar $R(s_0)$ in the following way [13]. The divergence of the generating vectors of the first cone is defined by

$$\rho_1 = m^a \bar{m}^b \nabla_a F_b.$$

where $F = F(\Omega, u, x^i)$ and $F = 0$ describes the null cone. Near $\mathcal{I}$ the function F can be written as

$$F = F^0(u, x^i) + \Omega F^1(u, x^i) + \mathcal{O}(\Omega^2).$$

where $F^0 = u - Z$. Then

$$\begin{aligned} \rho_1(s_0, \mathcal{I}^+) &= m^a \bar{m}^b \nabla_a (u - Z) \\ &= \rho_B - R(s_0). \end{aligned} \tag{23}$$

where ρ_B is the divergence of a Bondi congruence.

On the other hand, the Sachs-Penrose reciprocity theorem, states that:

Theorem: *Assume that X_1 and X_2 are two matrices whose elements are the tetrad components of the two complex deviation vectors associated with a null cone congruence with apex at a point p_1 and p_2 respectively, then*

$$X_1(\text{ at } p_2) = -X_2(\text{ at } p_1).$$

Hence using this theorem, we may assert that if the past null cone from $\mathcal{I}^+$ has a conjugate point at an affine distance $s = s*$ then the future null cone from $s*$ has

a conjugate point at $\mathcal{I}^+$. Then in the limit when $s_0 \to s*$, we find that $\rho_1(s_0, \mathcal{I}^+)$ goes to $+\infty$ and from(23) $R(s*) = -\infty\Box$

The first consequence of this lemma is that our coordinate system is well defined in the domain $R \in (-\infty, \infty)$, in other words from our coordinate system we can not detect the caustics that arise in the past null cones as we move into the space-time

The lemma is also useful to answer the question we possed before, namely, if there exist asymptotically flat space times that can be covered with a global canonical coordinate system constructed from a single function Z. Using proposition 4.4.5 [11] which states that any null congruence along a geodesic such that the affine length can be extended arbitrarily has a pair of conjugate points and lemma IV.1 we conclude that the coordinate system $(u, w, \bar{w}, R)$ derived from Z cannot cover a space-time except for Minkowski space. We conclude that

Proposition IV.2 *A single function $Z(x, \zeta, \bar{\zeta})$ cannot generate the conformal structure of a radiative space-time except for the Minkowski space.*

Is is easy to show that $\eth Z$ and $\bar{\eth} Z$ remain finite at a conjugate point. This follows from the fact that both are constants along a null geodesic.

It is also of interest to analize the behaviour of the conformal factor Ω, and $\Lambda = \eth^2 Z$ near a caustic since they generate the underlying metric of the space-time.

In a similar way than the Lemma above, we consider the future null cone with apex at s_0. The Sachs Theorem tells us that at $\mathcal{I}^+$

$$\sigma_1(s_0, \mathcal{I}^+) = m^a m^b \nabla_a l_b = \sigma_B - \Lambda(s_0). \tag{24}$$

where σ_B is the shear of a Bondi congruence.

If the future null cone of s_0 has a conjugate point at $\mathcal{I}^+$, then the shear $\sigma_1(s_0, \mathcal{I}^+)$ is either 0 or ∞. It then follows from the Sachs-Penrose reciprocity theorem and eq. (24) that $|\Lambda(s_0)|$ is either 0 or ∞ if the past null cone from $\mathcal{I}^+$ has a conjugate point at s_0.

It remains to consider the conformal factor

$$\Omega^2 = g^{01} := g^{ab} Z_{,a} \eth\bar{\eth} Z_{,b} = \frac{dR}{ds}.$$

Since $R(s)$ diverges as s approaches a conjugate point while the affine length is a smooth non-vanishing function along the null geodesic it follows that g^{01} also blows up at that point.

V. CONCLUSIONS

We have shown that our main variable can be regarded as the generating family $\widehat{Z}$ of a constrained Lagrange submanifold and that its level surfaces are constrained Legendre submanifolds that project down to past null cones from $\mathcal{I}^+$. Furthermore, we have demonstrated that for a generic space-time this Lagrange submanifold starts diffeomorphic to the configuration space but it develops caustic sets, points on $T^*(M)$ where its projections are caustic points on M. Thus, except for Minkowski space, a single function Z on configuration space does not give the conformal structure of the space-time. At the caustic points $\eth\bar{\eth}Z$ diverges. This means that the coordinate system constructed on the null cones is only locally defined but on the other hand one never sees the caustics since they are pushed out to $R = -\infty$.

Although the entire treatment so far has been kinematical we would like to think of our variable as coming from the solution of a set of field equations given on the space-time [3,4].

It is clear from the previous results that the solution of those field equations must have multiple valuedness in order to generate the multiple branches needed to construct the generating family $\widehat{Z}$ of the Lagrange submanifold. These solutions are defined in a 6-dimensional space, 4-spacetime coordinates and 2 parameters on the sphere, $(\zeta, \bar{\zeta})$. We demand the solution to be globally defined respect to the parameters $(\zeta, \bar{\zeta})$, that is, it shall be a piecewise smooth function on the sphere (it could be multiple valued but always finite on the sphere). In this sence we say that the solutions shall be **globally regular** on the space of parameters.

Alternatively, we could try to find field equations given on T^*M. In this case the solution would yield a global generating family $\widehat{Z}$ of a constrained Lagrange submanifold that coincides with Z in a neighbourhood of $\mathcal{I}^+$. This last approach

will be further explored.

Acknowledgments

We are indebted to Ted. E. Newman and Simonetta Frittelli for many enlightening conversations and suggestions. This research has been partially supported by AIT, CONICET and CONICOR.

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